# 1 No friction

The figure shows the cyclonic wind (azimuthal velocity, $v$) in the planetary boundary layer (the lowest kilometers) of a polar low. The wind varies from very strong close to the center, to zero at some radius and negative (anti-cyclonic) at larger radius. The variation is due to conservation of rotation (angular momentum, $M$). In the figure, $M$ is controlled by setting the latitude and the radius of zero cyclonic wind.

**Motivation:**

We assume that the polar low is in a steady state and axis-symmetric. We treat the boundary layer as a vertically homogeneous slab. This means that there is no variation in time, nor in the vertical and the azimuthal directions. The only variation is in the radial direction. With our assumptions the momentum equation for the azimuthal direction becomes

$$u\frac{\mathrm{d}v}{\mathrm{d}r} = -u(\frac{v}{r} + f) + F_a, \tag{1}$$

where $u$ is the velocity in the radial direction, $r$ is the radial co-ordinate with zero at the cyclone centre, $v$ is the velocity in the azimuthal direction, $f$ is the Coriolis parameter and $F_a$ is the friction in the azimuthal direction. For a frictionless flow $F_a = 0$. Dividing equation (1) with $u$ we get

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -(\frac{v}{r} + f), \tag{2}$$

with the solution

$$v = \frac{M}{r} - \frac{f}{2}r, \tag{3}$$

where $M$ is an integration constant. Note that we have no information about the radial velocity. We see that $M$ is identical to the angular momentum defined as

$$M \equiv rv + \frac{f}{2}r^2. \tag{4}$$

In other words, equation (3) tells us that the variation of azimuthal velocity along the radius is caused by the conservation of angular momentum. The figure shows the azimuthal velocity calculated with equation (3). Since $M$ is conserved, we may also define $M$ as

$$M \equiv \frac{f}{2} r_0^2, \tag{5}$$

where $r_0$ is the radius of zero azimuthal velocity. In the figure, $M$ is set by setting $r_0$ and the latitude, yielding $f$.

# 2 Adding friction

In the frictionless case the cyclonic wind is unrelated to the radial wind. Adding friction changes that. The friction leads to a continous loss of angular momentum. To maintain a steady state there must be a compensating radial transport of angular momentum.

Conservation of mass (and small variation of density) requires a lower radial velocity at larger radius, which means longer time for the friction to act. This leads to a vannishing gradient of the cyclonic wind at large radius and the wind never becomes anticyclonic.

**Motivation:**

Let us assume that the frictional acceleration in the azimuthal direction can be calculated as

$$F_a = -\frac{C_D}{h} U v, \tag{6}$$

where $C_D$ is a dimensionless drag coefficient, $h$ is the height of the boundary layer and $U = \sqrt{u^2 + v^2}$ is the magnitude of the (horizontal) wind. With this parametrisation equation (1) becomes

$$u \frac{\mathrm{d}v}{\mathrm{d}r} = -u\left(\frac{v}{r} + f\right) - \frac{C_D}{h} U v. \tag{7}$$

We want to divide equation (7) with $u$, but first we define a parameter, $A$,

$$A \equiv \frac{urh}{C_D}. \tag{8}$$

$A$ is the radial volumeflow, scaled with $2\pi C_D$. Note that convergence towards the cyclone center corresponds to negative values of $u$ and $A$. Dividing equation (7) with $u$, using equation (8), yields

$$\frac{\mathrm{d}v}{\mathrm{d}r} = -\frac{v}{r} - f - \frac{r}{A} U v. \tag{9}$$

At large radius it is reasonable to assume that both the gradient of the wind and the curvature of the wind are small. The asymptote for $v$ at large radii can hence be calculated from the balance of only the two last terms on the right hand side of equation (9). We further assume that the radial wind is small compared to the cyclonic wind ($U = |v|$). We get

$$|v| = \left(\frac{f|A|}{r}\right)^{\frac{1}{2}}, \tag{10}$$

where $v$ should have the opposite sign to $A$. We can find $v$ by using this asymptote[1] and integrating equation (9) numerically. At small radii we could guess that the friction term is little and that the solution is identical to equation (3). The angular momentum ($M_0$) of the small radius asymptote appears to be[2]

$$M_0 = \left(\frac{fA^2}{e^1}\right)^{1/3}, \tag{11}$$

$$\Longleftrightarrow$$

$$A = fr_0^3\left(\frac{e^1}{2}\right)^{1/2}, \tag{12}$$

where we have used equation (5). Note that the squareroot expression is approximately unity, so $A \simeq fr_0^3$. We find that in a boundary layer with friction the azimuthal velocity is primarily determined by $A$ and $f$. If $A$ is given, $C_D$ and $h$ only influnce $v$ where $U$ is not completely dominated by the $v$-component. The figure shows the asymptotes together with the result of numerically integrating equation (9).

# 3 Temperature

$$u\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} + v(\frac{v}{r} + f) + F_r. \tag{13}$$

[1]The result at smaller radius is actually not very sensitive to the starting value of $v$. In the figure it is possible to insert a $v$-value at large radius, instead of using equation (10).

[2]My conviction rest upon dimensional arguments and on numerical testing. I am sure there is a more rigid proof...

$$F_r = -\frac{C_D}{h}|u|u. \tag{14}$$

$$c_p u \frac{\mathrm{d}T}{\mathrm{d}r} = \frac{u}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} + J, \tag{15}$$

$$J = -c_p \frac{C_{HE}}{h}\, U\, \Delta T, \tag{16}$$

$$c_p \frac{\mathrm{d}T}{\mathrm{d}r} = -u\frac{\mathrm{d}u}{\mathrm{d}r} + v(\frac{v}{r} + f) - \frac{r}{A}|u|u^2 - \frac{r}{A}\, U\, c_p \Delta T. \tag{17}$$

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{1}{c_p \rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{r}{Q}\, U\, \Delta T. \tag{18}$$